Solid state systems for quantum information, Correction 10

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Exercise 1 : Two-qubit gates

We consider two transmon qubits, coupled via a capacitance C_0, see Fig. 1.

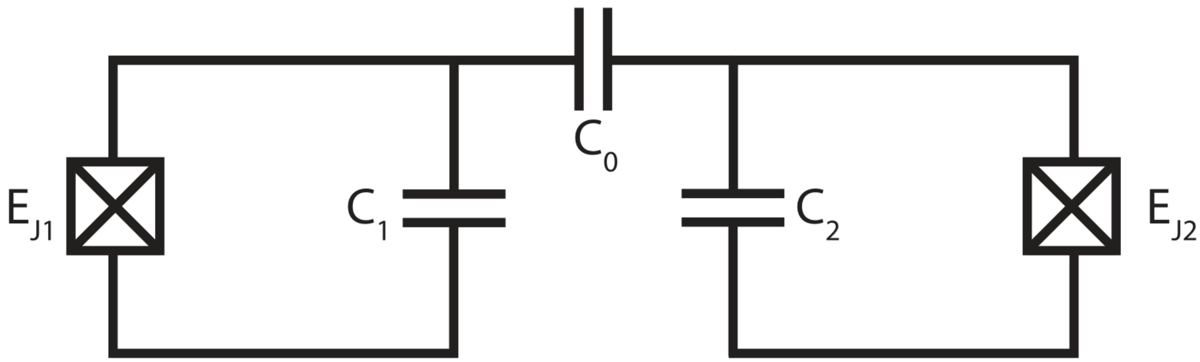

Figure 1: Electric circuit of two transmon qubits coupled via a capacitor C_0.

The effective two-qubit Hamiltonian describing the system reads

$$H/\hbar = \sum_{i=1}^{2} \left[\omega_i b_i^\dagger b_i + \frac{\alpha_i}{2} b_i^\dagger b_i^\dagger b_i b_i\right] + J(b_1^\dagger b_2 + b_2^\dagger b_1), \tag{1}$$

where ω_i is the frequency of the respective qubit, α_i is the anharmonicity, J is the coupling rate between two qubits and $b_i(b_i^\dagger)$ indicates the annihilation (creation) operator of the respective qubit.

1. Show, that the matrix representation of the Hamiltonian found in Eq. 1, with respect to the basis states $\{|00\rangle, |10\rangle, |01\rangle, |20\rangle, |11\rangle, |02\rangle\}$, is:

$$H/\hbar = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega_1 & J & 0 & 0 & 0 \\ 0 & J & \omega_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_1 + 2\omega_1 & \sqrt{2}J & 0 \\ 0 & 0 & 0 & \sqrt{2}J & \omega_1 + \omega_2 & \sqrt{2}J \\ 0 & 0 & 0 & 0 & \sqrt{2}J & \alpha_2 + 2\omega_2 \end{pmatrix} \tag{2}$$

   Use the fact that $b_1 |n, j\rangle = \sqrt{n}\,|(n-1), j\rangle$, and $b_2 |j, n\rangle = \sqrt{n}\,|j, (n-1)\rangle$, where $j \in \{0, 1, 2\}$.

2. Investigate how the system behaves if both qubits are on resonance, i.e. $\omega_1 = \omega_2 = 2\pi \cdot 6$ GHz. Use the prepared jupyter notebook to simulate the time evolution of the system for $J/2\pi = 8$ MHz, starting from $|\psi_0\rangle = |01\rangle$ and assume $\alpha_1 = \alpha_2 = -2\pi \cdot 300$ MHz. Plot the occupation probabilities of the states $|00\rangle, |01\rangle, |10\rangle$ and $|11\rangle$ for $0 < t < \frac{3\pi}{2J}$. What is the two-qubit state you obtain at time $t = \frac{\pi}{4J}$?

3. Let us now consider the case $\omega_1 = \omega_2 - \alpha_1 = 2\pi \cdot 6$ GHz for $J/2\pi = 8$ MHz. Plot the evolution of the occupation probabilities, starting from an initial state $|11\rangle$ for $0 < t < \frac{3\pi}{2J}$. After ehat time t_1 has the system returned back to its initial state $|11\rangle$? Why does the system return faster to the initial state $|11\rangle$ than it returned to the initial state $|01\rangle$ in 2.

4. Plot the time evolution of the system with $J/2\pi = 8$ MHz where $\omega_1 = \omega_2 - \alpha_1 - \delta = 2\pi \cdot 6$ MHz, for detunings $\delta/2\pi \in [-50, 50]$ MHz. For all detunings, find the time $t_i(\delta)$ you need to evolve the system in order to return to the state $|11\rangle$ (when starting in $|11\rangle$). Plot this time $t_i(\delta)$ vs the detuning.

Solution 1 :

1. To determine the matrix representation of the Hamiltonian H, we evaluate

$$\begin{aligned}\langle mn|H/\hbar|m'n'\rangle = \sum_{i=1}^{2} \left[\omega_i \left\langle mn \left| b_i^\dagger b_i \right| m'n' \right\rangle + \frac{\alpha_i}{2} \left\langle mn \left| b_i^\dagger b_i^\dagger b_i b_i \right| m'n' \right\rangle\right] \\ + J\left(\left\langle mn \left| b_1^\dagger b_2 \right| m'n' \right\rangle + \left\langle mn \left| b_2^\dagger b_1 \right| m'n' \right\rangle\right)\end{aligned} \tag{3}$$

Using the definition of the annihilation operators b_1, b_2

$$\begin{aligned} b_1|nj\rangle &= \sqrt{n}|(n-1)j\rangle, \text{ for } j \in \{0,1,2\} \\ b_2|jn\rangle &= \sqrt{n}|j(n-1)\rangle, \text{ for } j \in \{0,1,2\}, \end{aligned} \tag{4}$$

we look at each term individually:

$$\begin{aligned}\left\langle mn \left| b_1^\dagger b_1 \right| m'n' \right\rangle &= (b_1|mn\rangle)^\dagger b_1 |m'n'\rangle = (\sqrt{m}|(m-1)n\rangle)^\dagger \sqrt{m'}\,|(m'-1)\,n'\rangle \\ &= \sqrt{m}\sqrt{m'}\,\langle (m-1)n \mid (m'-1)\,n'\rangle = m\delta_{mm'}\delta_{nn'}\end{aligned} \tag{5}$$

where we use the Kronecker delta:

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \tag{6}$$

Similarly,

$$\langle mn|\, b_2^\dagger b_2 \,|m'n'\rangle = n\delta_{mm'}\delta_{nn'} \tag{7}$$

For the second (anharmonic) term, we get:

$$\begin{aligned}\left\langle mn \left| b_1^\dagger b_1^\dagger b_1 b_1 \right| m'n' \right\rangle &= (b_1 b_1|mn\rangle)^\dagger b_1 b_1 \,|m'n'\rangle = \left(b_1\sqrt{m}|(m-1)n\rangle\right)^\dagger b_1\sqrt{m'}\,|(m'-1)\,n'\rangle \\ &= (\sqrt{m-1}\sqrt{m}|(m-2)n\rangle)^\dagger \sqrt{m'-1}\sqrt{m'}\,|(m'-2)\,n'\rangle \\ &= \sqrt{m-1}\sqrt{m}\sqrt{m'-1}\sqrt{m'}\,\langle (m-2)n \mid (m'-2)\,n'\rangle \\ &= m(m-1)\delta_{mm'}\delta_{nn'}\end{aligned} \tag{8}$$

Similarly, we get

$$\langle mn| b_2^\dagger b_2^\dagger b_2 b_2 |m'n'\rangle = n\delta_{mm'}\delta_{nn'} \tag{9}$$

For the coupling terms, we obtain

$$\begin{aligned}\left\langle mn \left| b_1^\dagger b_2 \right| m'n' \right\rangle &= (b_1|mn\rangle)^\dagger\, b_2\, |m'n'\rangle = (\sqrt{m}|(m-1)n\rangle)^\dagger \sqrt{n'}\, |m'\,(n'-1)\rangle \\ &= \sqrt{m}\sqrt{n'}\,\langle (m-1)n \mid m'\,(n'-1)\rangle = \sqrt{m}\sqrt{n'}\delta_{(m-1)m'}\delta_{n(n'-1)}\end{aligned} \tag{10}$$

and

$$\left\langle mn \left| b_2^\dagger b_1 \right| m'n' \right\rangle = \sqrt{m'}\sqrt{n}\delta_{m(m'-1)}\delta_{(n-1)n'} \tag{11}$$

The Hamiltonian then reads

$$\begin{aligned}\langle mn|H/\hbar|m'n'\rangle &= \delta_{mm'}\delta_{nn'}\left(\omega_1 m + \omega_2 n + \frac{n_1}{2}m(m-1) + \frac{\alpha_2}{2}n(n-1)\right) \\ &\quad + J\sqrt{m}\sqrt{n'}\delta_{(m-1)m'}\delta_{n(n'-1)} + J\sqrt{m'}\sqrt{n}\delta_{m(m'-1)}\delta_{(n-1)n'}\end{aligned} \tag{12}$$

Next, we evaluate this equation for all basis states $\{|00\rangle, |10\rangle, |01\rangle, |20\rangle, |11\rangle, |02\rangle\}$. For the diagonal, i.e. for matrix elements $\langle mn| H/\hbar |mn\rangle$, we have to evaluate all terms with $\delta_{mm'}\delta_{nn'}$. We then obtain $\{0, \omega_1, \omega_2, \alpha_1{+}2\omega_1, \omega_1{+}\omega_2, \alpha_2{+}2\omega_2\}$ for $(m, n) \in \{|00\rangle, |10\rangle, |01\rangle, |20\rangle, |11\rangle, |02\rangle\}$. The only non-zero off diagonal terms originate from $\delta_{(m-1)m'}\delta_{n(n'-1)}$ and $\delta_{m(m'-1)}\delta_{(n-1)n'}$, which are only non-zero for the terms $\langle 01| H/\hbar |10\rangle$, $\langle 10| H/\hbar |01\rangle$, $\langle 20| H/\hbar |11\rangle$, $\langle 11| H/\hbar |20\rangle$, $\langle 02| H/\hbar |11\rangle$ and $\langle 11| H/\hbar |02\rangle$.

Evaluating the remaining terms then results in the given matrix representation:

$$H/\hbar = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega_1 & J & 0 & 0 & 0 \\ 0 & J & \omega_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_1 + 2\omega_1 & \sqrt{2}J & 0 \\ 0 & 0 & 0 & \sqrt{2}J & \omega_1 + \omega_2 & \sqrt{2}J \\ 0 & 0 & 0 & 0 & \sqrt{2}J & \alpha_2 + 2\omega_2 \end{pmatrix} \tag{13}$$

2. When both qubits are on resonance, i.e. $\omega_1 = \omega_2$, we observe Rabi oscillations with rate J between the states $|01\rangle$ and $|10\rangle$, see Fig.2. After the time $t = \frac{\pi}{4J}$, we obtain the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|10\rangle + i\,|01\rangle). \tag{14}$$

The state $|\psi\rangle$ is maximally entangled and equivalent to a Bell state (up to local rotations). Thus, if we turn off the interaction after $t = \frac{\pi}{4J}$, we would have implemented a $\sqrt{i\text{SWAP}}$ gate. Remark: Single-qubit gates and a $\sqrt{i\text{SWAP}}$ gate form a universal set of gates.

3. For $\omega_1 = \omega_2 - \alpha_1$, the states $|11\rangle$ with frequency $(\omega_1 + \omega_2)/2\pi$, and state $|20\rangle$, with frequency $2(\omega_1 + \alpha_1)/2\pi = (2\omega_1 + (\omega_2 - \omega_1))/2\pi = (\omega_1 + \omega_2)/2\pi$, are on resonance. Thus, we observe Rabi oscillations between the states $|11\rangle$ and $|20\rangle$, see Fig. 3. After the time $t = \frac{\pi J}{\sqrt{2}} = 44.2$ ns, the system has returned to its initial state $|11\rangle$. Thus, we have Rabi rate of $\sqrt{2}J$, which is $\sqrt{2}$ times as fast as the Rabi oscillations shown in Fig. 2. This is due to the larger coupling rate $\sqrt{2}J$, between the states $|20\rangle$ and $|11\rangle$, see elements $\langle 20| H/\hbar |11\rangle$ and $\langle 11| H/\hbar |20\rangle$ in Eq. (2).

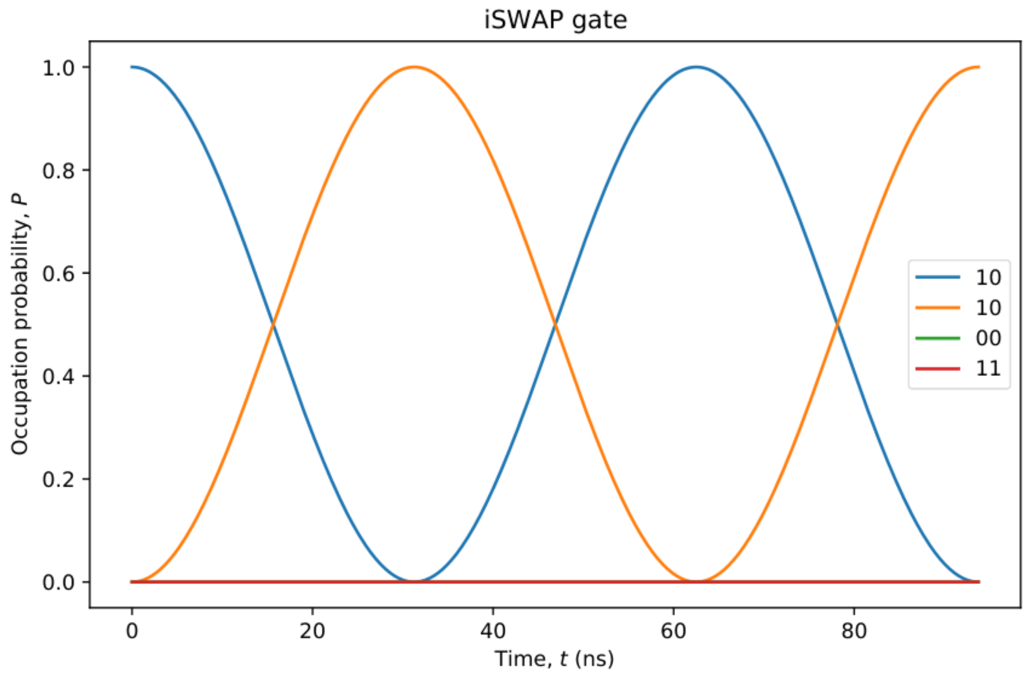

Figure 2: Occupation probabilities P vs time for the case $\omega_1 = \omega_2$.

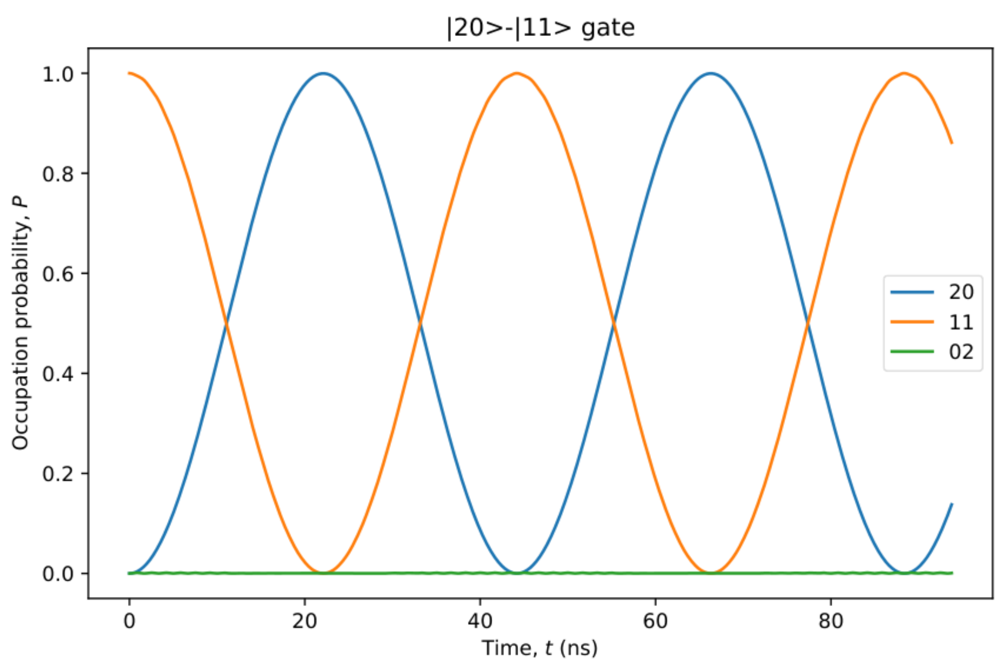

Figure 3: Occupation probabilities P vs time for the case $\omega_1 = \omega_2 - \alpha_1$.

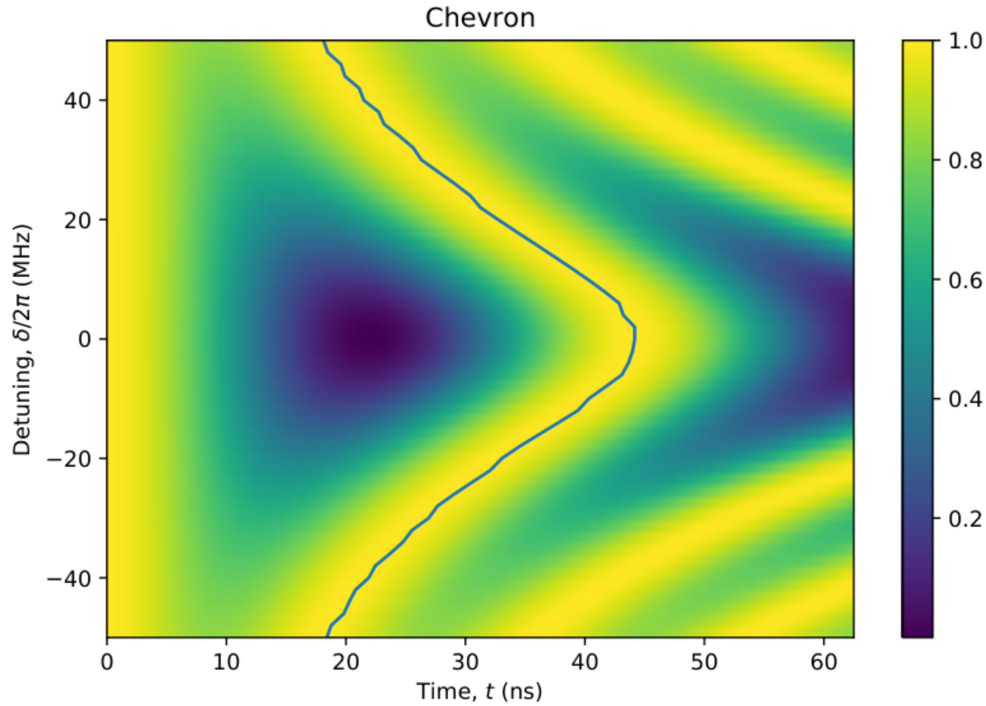

Figure 4: Occupation probabilities P of the state $|11\rangle$ vs time t and detuning δ for $\omega_1 = \omega_2 - \alpha_1 + \delta$. The blue line indicated the times $t_i(\delta)$ at which the system has returned to its initial state $|11\rangle$ for each detuning.

4. When investigating the time evolution of the initial state $|11\rangle$ for different detunings δ in $\omega_1 = \omega_2 - \alpha_1 + \delta$, we notice that the rate of Rabi oscillations increases with increasing δ, while the amplitude of the Rabi oscillations decreases, see Fig. 4. For $|\delta| > 0$, we therefore never fully occupy state $|20\rangle$. The time $t_i(\delta)$ needed to return into the initial state $|11\rangle$ depends strongly on the chosen detuning, see also Fig. 4. The observed pattern is called "Chevron" and is typically observed when looking at a time evolution of two coupled modes around a resonance.

Exercise 2 : Virtual Photon Exchange

When two atoms are coupled to the same cavity, you have two Jaynes-Cummings systems. The full Hamiltonian is given by:

$$\hat{H} = \hat{H}_0 + \hat{H}_I \tag{15}$$

with

$$\hat{H}_0 = \hbar\omega_c \left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right) + \frac{\hbar\omega_a}{2}\hat{\sigma}_{z,1} + \frac{\hbar\omega_a}{2}\hat{\sigma}_{z,2} \tag{16}$$

and

$$\hat{H}_I = \hbar g(\hat{a}^\dagger \hat{\sigma}_{-,1} + \hat{a}\hat{\sigma}_{+,1}) + \hbar g(\hat{a}^\dagger \hat{\sigma}_{-,2} + \hat{a}\hat{\sigma}_{+,2}), \tag{17}$$

where ω_c is the frequency of cavity photons, $\hat{a}^\dagger$ and $\hat{a}$ create and anihilate cavity photons, ω_a is the frequency spacing of the atomic levels, g quantifies the coupling strength, and $\hat{\sigma}_{\pm,i} = \frac{1}{2}(\hat{\sigma}_{x,i} \pm i\hat{\sigma}_{y,i})$ for atom i. Our goal is to rewrite this Hamiltonian as the following (notice that, in the end, we do not consider the cavity anymore):

$$\hat{H} \approx \frac{\hbar\omega_a}{2}\hat{\sigma}_{z,1} + \frac{\hbar\omega_a}{2}\hat{\sigma}_{z,2} + J(\hat{\sigma}_{-,1}\hat{\sigma}_{+,2} + \hat{\sigma}_{+,1}\hat{\sigma}_{-,2}) \tag{18}$$

1. We will in the following assume that $g \ll |\Delta| = |\omega_a - \omega_c|$. Why do you expect that this assumption will allow us to eventually not consider the cavity in the Hamiltonian?

2. Show that the commutator

   $$[(\hat{a}\hat{\sigma}_{+,i} - \hat{a}^\dagger\hat{\sigma}_{-,i}), \hat{H}] = -\hbar\Delta(\hat{a}^\dagger\hat{\sigma}_{-,i} + \hat{a}\hat{\sigma}_{+,i}) + \hbar g(2\hat{a}^\dagger\hat{a}\hat{\sigma}_z + \hat{\sigma}_{z,i}) + \hbar g(\hat{\sigma}_{-,i}\hat{\sigma}_{+,j} + \hat{\sigma}_{+,i}\hat{\sigma}_{-,j}) \tag{19}$$

   for any i (with $j \neq i$).

3. The idea is now to perform a unitary transformation $H \to UHU^\dagger$ with the unitary $U = \exp\left(\sum_i \frac{g}{\Delta}(\hat{a}\hat{\sigma}_{+,i} - \hat{a}^\dagger\hat{\sigma}_{-,i})\right)$. It turns out, that this unitary transformation approximately diagonalizes the initial Hamiltonian.

   You can now use the "Baker-Campbell-Hausdorf" formula:

   $$e^A B e^{-A} = B + [A, B] + \frac{1}{2}[A, [A, B]] + \cdots + \frac{1}{n!}\overbrace{[A, [A, \ldots [A}^{nA's}, B]\ldots] + \ldots \tag{20}$$

   Calculate the transformed Hamiltonian up to second order in (g/Δ)

   Hint: You will need some terms from the $\frac{1}{2}[A, [A, B]]$-terms.

4. The resulting Hamiltonian now contains both cavity and qubit terms. Bring the transformed Hamiltonian to the form in Eq. 18 by assuming that the cavity is in the ground state at all times. What is J? Interpret the fact that the qubit interact while we explicitly assumed the cavity to be in the ground state.

5. Since the qubits are on resonance, it is enough to just consider the Hamiltonian $H = J(\hat{\sigma}_{-,1}\hat{\sigma}_{+,2} + \hat{\sigma}_{+,1}\hat{\sigma}_{-,2})$. If the qubits start in state $|1\rangle|0\rangle$, what is the state after $t = \pi/J$ and after $t = \pi/(2J)$?

Solution 2 :

1. Since the cavity is at a very different frequency, the transfer of excitations form and to the cavity will not be energetically allowed.

2. In order to show this relation we make use of the following commutation relations $\left[\hat{a}^\dagger, \hat{a}\right] = 1$, $[\sigma_+, \sigma_-] = \sigma_z$ and $[\sigma_\pm, \sigma_z] = \mp 2\sigma_\pm$ and the following properties, $[AB, C] = A[B, C] + [A, C]B$ and $[A, BC] = B[A, C] + [A, B]C$
Let's first start this derivation with the self-energy term of the Hamiltonian, $\hat{H}_0$:

$$\left[\left(\hat{a}\hat{\sigma}_{+,i} - \hat{a}^\dagger\hat{\sigma}_{-,i}\right), \hat{H}_0\right] = \hbar\omega_c[\hat{a}\hat{\sigma}_{+,i}, \hat{a}^\dagger\hat{a}] + \frac{\hbar\omega_a}{2}[\hat{a}\hat{\sigma}_{+,i}, \hat{\sigma}_{z,i}] + \frac{\hbar\omega_a}{2}[\hat{a}\hat{\sigma}_{+,i}, \hat{\sigma}_{z,j}] - \tag{21}$$

$$\left(\hbar\omega_c[\hat{a}^\dagger\hat{\sigma}_{-,i}, \hat{a}^\dagger\hat{a}] + \frac{\hbar\omega_a}{2}[\hat{a}^\dagger\hat{\sigma}_{-,i}, \hat{\sigma}_{z,i}] + \frac{\hbar\omega_a}{2}[\hat{a}^\dagger\hat{\sigma}_{-,i}, \hat{\sigma}_{z,j}]\right) \tag{22}$$

$$= \hbar\omega_c(\hat{a}^\dagger\hat{\sigma}_{-,i} + \hat{a}\hat{\sigma}_{+,i}) - \hbar\omega_a(\hat{a}^\dagger\hat{\sigma}_{-,i} + \hat{a}\hat{\sigma}_{+,i}) \tag{23}$$

$$= -\hbar\Delta\left(\hat{a}^\dagger\sigma_{-,i} + \hat{a}\hat{\sigma}_{-,i}\right). \tag{24}$$

Now let's look at the interaction term:

$$\left[\left(\hat{a}\hat{\sigma}_{+,i} - \hat{a}^\dagger\hat{\sigma}_{-,i}\right), \hat{H}_I\right] = \hbar g(\hat{a}\hat{a}^\dagger\sigma_{z,i} + \hat{\sigma}_{-,i}\hat{\sigma}_{+,i} + \hat{\sigma}_{-,j}\hat{\sigma}_{+,i} + \hat{\sigma}_{+,j}\hat{\sigma}_{-,i} + \hat{\sigma}_{+,i}\hat{\sigma}_{-,i}) \tag{25}$$

$$= \hbar g((2\hat{a}^\dagger\hat{a} + 1)\sigma_{z,i} + \hat{\sigma}_{-,j}\hat{\sigma}_{+,i} + \hat{\sigma}_{+,j}\hat{\sigma}_{-,i}). \tag{26}$$

This has been obtained decomposing the commutation relation in the same way as for the non-interacting term. Finally we obtain:

$$[(\hat{a}\hat{\sigma}_{+,i} - \hat{a}^\dagger\hat{\sigma}_{-,i}), \hat{H}] = -\hbar\Delta(\hat{a}^\dagger\hat{\sigma}_{-,i} + \hat{a}\hat{\sigma}_{+,i}) + \hbar g(2\hat{a}^\dagger\hat{a}\hat{\sigma}_z + \hat{\sigma}_{z,i}) + \hbar g(\hat{\sigma}_{-,i}\hat{\sigma}_{+,j} + \hat{\sigma}_{+,i}\hat{\sigma}_{-,j}). \tag{27}$$

3. The first two terms of the BCH give us the following using the results from 2.

$$H - H_I + \hbar\frac{g^2}{\Delta}\sum_i\left(2\hat{a}^\dagger\hat{a}\hat{\sigma}_{z,i} + \hat{\sigma}_{z,i}\right) + 2\hbar g\left(\hat{\sigma}_{-,i}\hat{\sigma}_{+,j} + \hat{\sigma}_{+,i}\hat{\sigma}_{-,j}\right). \tag{28}$$

This is almost the end as the third term has prefactor g^2/Δ^2 which mean we should neglect this term. Notice, however, that the first term in Eq. (5) makes this prefactor g^2/Δ, so we keep this term. The term is up to a prefactor the same as H_I, so we know the commutator already as it is the same as the last two terms of Eq. (5), so the final result is

$$H_0 + \hbar\frac{g^2}{\Delta}\sum_i\left(\hat{a}^\dagger\hat{a}\hat{\sigma}_{z,i} + \hat{\sigma}_{z,i}/2\right) + \hbar\frac{g^2}{\Delta}\left(\hat{\sigma}_{-,i}\hat{\sigma}_{+,j} + \hat{\sigma}_{+,i}\hat{\sigma}_{-,j}\right). \tag{29}$$

4. We know simply remove the $\hat{a}^\dagger\hat{a}$-term, as that is the only cavity operator in the transformed Hamiltonian and $\langle 0|\hat{a}^\dagger\hat{a}|0\rangle = 0$. We get the target Hamiltonian by using $\tilde{\omega}_a = \omega_a + g^2/\Delta$ and $J = \hbar g^2/\Delta$.

5. After $t = \pi/J$ we get the state $-|1\rangle|0\rangle$. For $t = \pi/(2J)$, we have $i|1\rangle|0\rangle$.

Exercise 3 : Superconducting coupling bus

This problem is about the paper "Coupling superconducting qubits via a cavity bus" by J. Majer et al (2007). You can find the paper on Moodle.
In the previous problem, you derived the main formula of this paper. Here, we will discuss some experimental aspects of this work. This paper was a milestone in the field of superconducting qubits as it was the first to demonstrate that superconducting qubits can interact through coupling bus.

1. In this paper, what is the qubit-cavity coupling strength for the two qubits ?
2. If we consider one of the qubits detuned from the other, we can write the Hamiltonian

   $$\hat{H} \approx \frac{\hbar\delta}{2}\hat{\sigma}_{z,1} + J(\hat{\sigma}_{-,1}\hat{\sigma}_{+,2} + \hat{\sigma}_{+,1}\hat{\sigma}_{-,2}) \tag{30}$$

   Find the eigenstates and eigenenergies for this Hamiltonian, first for $\delta = 0$ and then for any δ?

   Hint: You will have $|0\rangle|1\rangle$ and $|1\rangle|0\rangle$ as bare eigenstates and then two "dressed" eigenstates as linear combinations of $|0\rangle|1\rangle$ and $|1\rangle|0\rangle$.
3. What is the minimum energy difference between the two dressed eigenstates? How can you see this value in Figure 2?
4. Explain, conceptually, all additional features you see in Figure 2.
5. The paper mentions the a.c. Stark terms with the strength χ_i. Explain the effect of these terms when adding photons to the cavity.
6. Explain how the a.c. Stark effect was used to change the effective Hamiltonian in Figure 4.
7. Explain the oscillations in Figure 4.
8. The oscillation frequency follow a parabola in Figure 4 (d). Use the results of problem 3. to explain the parabolic shape.
9. Bonus question: While the coupling through a bus resonators, as in this paper, is widely used today in superconducting qubit labs at ETH, Google, Delft and many more, it is now possible to tune the frequency of qubits directly instead of using the a.c. Stark shift. What is the downsides of the a.c. Stark shift methods as demonstrated in this paper?

Solution 3 :

1. It is more or less the same for both qubits and equal to $g/2\pi = 105$ MHz.

2. We need to find the eigenstates and eigenvalues of the matrix:

$$\begin{pmatrix} \delta/2 & J \\ J & -\delta/2 \end{pmatrix}, \tag{31}$$

which is in the basis $|10\rangle, |01\rangle$. After diagonalizing this matrix, we get the eigenstates $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ and $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ with eigenvalues $\pm\sqrt{(\delta/2)^2 + J^2}$.

3. The difference is 2J, you see it as the difference at the avoided crossing in Fig. 2b.

4. In Fig.2a we see that the frequency of the two qubits changes with applied magnetic field (dashed lines, green and red). We then see avoided crossing between the cavity and the qubits individually in the bottom part when the qubit frequency matches the cavity. Lastly, we see qubit-qubit dressed states when the two qubits match in frequency, as also reported in the zoom in of Fig.2b. Here, we see also a dark state in correspondence of the symmetric state $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. The state is dark because the excitation between the two qubits mediated by the cavity is antisymmetric (due to the fact that the two qubits are placed at the opposite sides of a $\lambda/2$ cavity) and therefore it cannot excite the symmetric state $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$.

5. The a.c. Stark shift is the photon number-dependence shift of the resonance frequency of the qubit. The higher the power of the drive, the larger the number of photons and therefore the larger the frequency shift. The authors use the Stark shift to bring the qubits in- and out-of resonance each other.

6. The dispersive shifts χ_1 and χ_2 are slightly different for qubits 1 and 2. The $a^\dagger a$ term changes the amplitude of χ_1 and χ_2, resulting in an effective way to change the frequency of the qubits, for instance to bring them in- and out-of resonance each other.

7. When the Stark drive brings the two qubits in resonance, J is maximum and with interaction ON the $|01\rangle$ and $|10\rangle$ states are not eigenstates of the systems, therefore oscillations between the two can be observed.

8. By looking at the energy difference between the eigenvalues $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ and $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$, we get a parabolic behaviour as a function of δ.

9. The a.c. Stark shift is used to tune the qubits frequencies very fast. However, the idea of putting many photons inside the cavity at a certain rate, also implies that these many photons will leave the cavity fast. This is not desirable, because all these photons can decrease the qubits coherence.